BC Calculus
Practice Test: Series of Constants

Show all work for the following problems. No calculators!

1. Identify each sequence as convergent or divergent. If the sequence converges, find the limit.

a. $\left\{\frac{n}{1+\sqrt{n}}\right\}$ div

$\lim_{n\to\infty} \frac{n}{1+\sqrt{n}} = \infty$

b. $\left\{\cos\frac{1}{n}\right\}$ conv. to 1

$\lim_{n\to\infty} \cos\frac{1}{n} = 1$

c. $\left\{\frac{n!}{3^n}\right\}$ div

$\lim_{n\to\infty} \frac{n!}{3^n} = \infty$

d. $\left\{\left(1+\frac{1}{2n}\right)^{3n}\right\}$ conv to $e^{3/2}$

$\lim_{n\to\infty} \left(\left(1+\frac{\frac{1}{2}}{n}\right)^n\right)^3 = \left(e^{1/2}\right)^3$

2. Determine whether each series is convergent or divergent. If the series converges, find the sum.

a. $\sum_{n=2}^{\infty}\left(\frac{e}{4}\right)^n$ Inf geom series

$r = \frac{e}{4} \to$ conv. to $\frac{\frac{e^2}{16}}{1-\frac{e}{4}} \cdot \frac{16}{16} = \frac{e^2}{16-4e}$

$\lim_{n\to\infty} a_n = 0$

b. $\sum_{n=1}^{\infty}\frac{2}{n^2+4n+3} = \sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+3}\right)$ conv to $\frac{1}{2}+\frac{1}{3} = \frac{5}{6}$

$\frac{2}{(n+3)(n+1)} = \frac{A}{n+3} + \frac{B}{n+1}$

$2 = A(n+1) + B(n+3)$

$n=-1$: $2 = 2B$, $1 = B$

$n=-3$: $2 = -2A$, $-1 = A$

Does not fit

$S_1 = \frac{1}{2} - \frac{1}{4}$

$S_2 = \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5}$

$S_3 = \frac{1}{2} - \cancel{\frac{1}{4}} + \frac{1}{3} - \frac{1}{5} + \cancel{\frac{1}{4}} - \frac{1}{6}$

$S_4 = \frac{1}{2} + \frac{1}{3} - \cancel{\frac{1}{5}} - \frac{1}{6} + \cancel{\frac{1}{5}} - \frac{1}{7}$

$S_5 = \frac{1}{2} + \frac{1}{3} - \cancel{\frac{1}{6}} - \frac{1}{7} + \cancel{\frac{1}{6}} - \frac{1}{8}$

$S_n = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$

PMI

① $n=2 \Rightarrow S_2 = \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5}$ ✓

② $S_{n+1} = S_n + a_{n+1}$

$\frac{1}{2} + \frac{1}{3} - \frac{1}{n+3} - \frac{1}{n+4} = \frac{1}{2} + \frac{1}{3} - \cancel{\frac{1}{n+2}} - \frac{1}{n+3} + \cancel{\frac{1}{n+2}} - \frac{1}{n+4}$ ✓

$\lim_{n\to\infty} S_n = \frac{1}{2} + \frac{1}{3} - 0 - 0$

c. $\sum_{n=1}^{\infty} \tan^{-1} n$ div by n^{th} term test

$\lim_{n \to \infty} \tan^{-1} n = \pi/2$

3. Classify each series as convergent or divergent. Justify your answers.

a. $\sum_{n=1}^{\infty} \frac{1}{n^3}$ p-series

$p=3$

conv.

b. $\sum_{n=1}^{\infty} \left(\frac{\cos 2}{3}\right)^n$ Inf geom series $\to$ conv.

$r = \frac{\cos 2}{3}$

$-1 < \cos 2 < 0$

so $|r| < 1$

c. $\sum_{n=3}^{\infty} \frac{5}{n-2}$ harmonic all positive terms $\rightarrow$ div.

d. $\sum_{n=1}^{\infty} \frac{1}{n^2-4n-5}$ conv. by Limit Comparison

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ p-series $p=2$ conv.

$$\lim_{n\to\infty} \frac{\frac{1}{n^2-4n-5}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-4n-5} = 1$$ is finite and non zero

e. $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^4}$ conv. by Direct Comparison

$0 \leq |\sin n| \leq 1$

$0 \leq \frac{|\sin n|}{n^4} \leq \frac{1}{n^4}$

$\sum_{n=1}^{\infty} \frac{1}{n^4}$ p-series $p=4$ conv.

f. $\sum_{n=1}^{\infty} \frac{1}{n!}$ Conv. absolutely by Ratio Test

$$\lim_{n\to\infty} \left| \frac{1}{(n+1)!} \cdot \frac{n!}{1} \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

g. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ Conv. by Integral Test

$f(x) = \frac{1}{x(\ln x)^2}$

$f(x) > 0$ since $x \geq 2$

$f(x)$ is Cont. if $n \neq 0$ or 1

$\therefore (x+1)(\ln(x+1))^2 > x(\ln x)^2$

so $\frac{1}{(x+1)(\ln(x+1))^2} < \frac{1}{x(\ln x)^2}$

f dec

$$\lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u^2}\,du$$

$$\lim_{b\to\infty} -u^{-1} \Big|_{\ln 2}^{\ln b}$$

$$\lim_{b\to\infty} -\left(\frac{1}{\ln b} - \frac{1}{\ln 2}\right)$$

$$\frac{1}{\ln 2}$$

4. Answer the following questions for the function $f(x)=\sum_{n=0}^{\infty} x^n$. JUSTIFY YOUR ANSWERS.

a. If possible, find $f\left(-\frac{3}{4}\right)$.

$\sum_{n=0}^{\infty}\left(-\frac{3}{4}\right)^n$ inf. geom. $r=-3/4$ conv. to $\frac{1}{1+\frac{3}{4}} = \frac{1}{\frac{7}{4}} = \frac{4}{7}$

b. For what value(s) of x will f(x) converge?

$\sum_{n=0}^{\infty} x^n$ is inf. geom $\Rightarrow$ $|x|<1$ makes series conv.

c. Write a formula for f(x) that does NOT involve summation notation.

If $|x|<1$, $f(x)=\frac{1}{1-x}$

5. Answer the following questions about the function $g(x)=\sum_{n=1}^{\infty}(\sin x)^n$. JUSTIFY YOUR ANSWERS.

a. If possible, find $g\left(\frac{\pi}{4}\right)$.

$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n$ inf geom $r=\frac{1}{\sqrt{2}}$ conv. to $\frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{1}{\sqrt{2}-1}$

b. Write a formula for g(x) that does NOT involve summation notation.

$\frac{\sin x}{1-\sin x}$ if $|\sin x|<1$

c. For what value(s) of x will g(x) converge?

If $|\sin x|<1$, inf. geom series will conv.

so $\sin x \neq \pm 1$

$x \neq \frac{(2n+1)\pi}{2}$

BC Calculus I Pledge________________

Quiz: Convergence Tests (20 points)

Complete each of the following problems. *Justify your answers in order to receive credit*. Each blank is worth 2 points.

1. Given the following sequences, identify as convergent or divergent. If convergent, give the limit.

Conv. to $e^{1/4}$ a. $\left\{\left(1+\frac{1}{4n}\right)^n\right\}$ $\lim_{n\to\infty}\left(1+\frac{\frac{1}{4}}{n}\right)^n = e^{1/4}$

Conv to 2 b. $\left\{2+\frac{1}{n!}\right\}$ $\lim_{n\to\infty}\left(2+\frac{1}{n!}\right) = 2$

2. Determine if the following series converge or diverge. If the series converges, find the sum.

Conv to $\frac{7}{2}$ by inf geom series test a. $\sum_{n=0}^{\infty}\frac{5^n}{7^n}$ inf geom series $r=\frac{5}{7}$

$$\frac{1}{1-\frac{5}{7}} = \frac{1}{\frac{2}{7}}$$

div. by nth term b. $\sum_{n=2}^{\infty}3$ $\lim_{n\to\infty} 3 = 3$

Conv to 1 by Sequence of Partial Sums c. $\sum_{n=1}^{\infty}\frac{1}{n^2+n} = \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

$\frac{A}{n}+\frac{B}{n+1}=\frac{1}{n^2+n}$ $A(n+1)+Bn=1$

$n=-1$: $B=-1$ $n=0$: $A=1$

$S_1 = 1-\frac{1}{2}$

$S_2 = 1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3} = 1-\frac{1}{3}$

$S_3 = 1-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}$

$S_n = 1-\frac{1}{n+1}$

$\lim_{n\to\infty} S_n = 1$

1 $S_1 = 1-\frac{1}{1+1} = 1-\frac{1}{2}$

2 $S_{n+1} = S_n + a_{n+1}$

$1-\frac{1}{(n+1)+1} = 1-\frac{1}{n+1}+\frac{1}{n+1}-\frac{1}{n+2}$

$1-\frac{1}{n+2} = 1-\frac{1}{n+2}$

3. Identify each series as convergent or divergent.

conv. abs by Ratio Test

a. $\sum_{n=1}^{\infty} \frac{n^7}{9^n}$

$\lim_{n\to\infty} \frac{n^7}{9^n} = 0$

$\lim_{n\to\infty} \left| \frac{(n+1)^7}{9^{n+1}} \cdot \frac{9^n}{n^7} \right|$

$= \lim_{n\to\infty} \frac{1}{9} \left(\frac{(n+1)^7}{n^7} \right)$

$= \frac{1}{9} \cdot 1 \quad < 1$

conv. by Limit Comparison

b. $\sum_{n=1}^{\infty} \frac{5}{\sqrt[5]{n^6}}$

$\lim_{n\to\infty} \frac{5}{n^{6/5}} = 0$

$\sum_{n=1}^{\infty} \frac{1}{n^{6/5}}$ = p-series $p = 6/5$ conv.

$\lim_{n\to\infty} \frac{\frac{5}{n^{6/5}}}{\frac{1}{n^{6/5}}} = 5$

div by nth term test

c. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

$\lim_{n\to\infty} \frac{(2n)!}{n! \cdot n!} = \lim_{n\to\infty} \frac{(2n)(2n-1)(2n-2)\cdots(\cancel{2n-n})}{n(n-1)(n-2)\cdots \cdot \cancel{n}}$

$= \infty$

conv. by Limit Comp.

d. $\sum_{n=1}^{\infty} \frac{3n^3+3}{n^4+2n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ div p-series $p \le 1$

$\lim_{n\to\infty} \frac{\frac{3n^3+3}{n^4+2n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{3n^4+3}{n^4+2n^2} = 3$

4. Identify each series as conditionally convergent, absolutely convergent, or divergent.

div by nth term test

a. $\sum_{n=1}^{\infty} \left(1+\frac{2}{n}\right)^n$

$\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n = e^2$

Conv. abs. b. $\sum_{n=1}^{\infty}(-1)^n \frac{\sin n}{n^5}$

$$\lim_{n\to\infty} \frac{(-1)^n \sin n}{n^5} = 0$$

$$0 \le |\sin n| \le 1$$

$$0 \le \frac{|\sin n|}{n^5} \le \frac{1}{n^5}$$

$$0 \le \sum_{n=1}^{\infty} \frac{|\sin n|}{n^5} \le \sum_{n=1}^{\infty} \frac{1}{n^5}$$

conv. p-series $p=5$

$$\rightarrow \sum_{n=1} \frac{|\sin n|}{n^5} \text{ conv.}$$

by Direct Comparison

conditionally conv. c. $\sum_{n=3}^{\infty}(-1)^n \frac{1}{n+1}$

$$\lim_{n\to\infty} \frac{(-1)^n}{n+1} = 0$$

$\sum_{n=3}^{\infty} \frac{1}{n+1}$ harmonic, all positive terms $\Rightarrow$ div.

$\sum_{n=3}^{\infty} \frac{(-1)^n}{n+1}$ harmonic, alternating $\Rightarrow$ conv.