BC Calculus
Rubrics
FR 9 & 10

Question 1

$\frac{20}{1+x^2} = 2$ when $x = \pm 3$	1 : correct limits in an integral in (a), (b), or (c)
(a) Area $= \int_{-3}^{3}\left(\frac{20}{1+x^2} - 2\right)dx = 37.961$ or 37.962	$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(b) Volume $= \pi\int_{-3}^{3}\left(\left(\frac{20}{1+x^2}\right)^2 - 2^2\right)dx = 1871.190$	$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(c) Volume $= \frac{\pi}{2}\int_{-3}^{3}\left(\frac{1}{2}\left(\frac{20}{1+x^2} - 2\right)\right)^2 dx$ $= \frac{\pi}{8}\int_{-3}^{3}\left(\frac{20}{1+x^2} - 2\right)^2 dx = 174.268$	$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

Question 2

(a) $\int_0^7 f(t)\,dt = 8264$ gallons

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) The amount of water in the tank is decreasing on the intervals $0 \le t \le 1.617$ and $3 \le t \le 5.076$ because $f(t) < g(t)$ for $0 \le t < 1.617$ and $3 < t < 5.076$.

$2 : \begin{cases} 1 : \text{intervals} \\ 1 : \text{reason} \end{cases}$

(c) Since $f(t) - g(t)$ changes sign from positive to negative only at $t = 3$, the candidates for the absolute maximum are at $t = 0,\ 3,$ and 7.

t (hours)	gallons of water
0	5000
3	$5000 + \int_0^3 f(t)\,dt - 250(3) = 5126.591$
7	$5126.591 + \int_3^7 f(t)\,dt - 2000(4) = 4513.807$

The amount of water in the tank is greatest at 3 hours. At that time, the amount of water in the tank, rounded to the nearest gallon, is 5127 gallons.

$5 : \begin{cases} 1 : \text{identifies } t = 3 \text{ as a candidate} \\ 1 : \text{integrand} \\ 1 : \text{amount of water at } t = 3 \\ 1 : \text{amount of water at } t = 7 \\ 1 : \text{conclusion} \end{cases}$

Question 3

(a) Area $= \frac{2}{3}\pi(2)^2 + \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(3+2\cos\theta)^2\,d\theta$
$= 10.370$

$$4: \begin{cases} 1: \text{area of circular sector} \\ 2: \text{integral for section of limaçon} \\ \quad 1: \text{integrand} \\ \quad 1: \text{limits and constant} \\ 1: \text{answer} \end{cases}$$

(b) $\left.\frac{dr}{dt}\right|_{\theta=\pi/3} = \left.\frac{dr}{d\theta}\right|_{\theta=\pi/3} = -1.732$

The particle is moving closer to the origin, since $\frac{dr}{dt} < 0$ and $r > 0$ when $\theta = \frac{\pi}{3}$.

$$2: \begin{cases} 1: \left.\frac{dr}{dt}\right|_{\theta=\pi/3} \\ 1: \text{interpretation} \end{cases}$$

(c) $y = r\sin\theta = (3+2\cos\theta)\sin\theta$
$\left.\frac{dy}{dt}\right|_{\theta=\pi/3} = \left.\frac{dy}{d\theta}\right|_{\theta=\pi/3} = 0.5$

The particle is moving away from the x-axis, since $\frac{dy}{dt} > 0$ and $y > 0$ when $\theta = \frac{\pi}{3}$.

$$3: \begin{cases} 1: \text{expression for } y \text{ in terms of } \theta \\ 1: \left.\frac{dy}{dt}\right|_{\theta=\pi/3} \\ 1: \text{interpretation} \end{cases}$$

Question 4

(a) $f'(e) = e^2$

An equation for the line tangent to the graph of f at the point $(e, 2)$ is $y - 2 = e^2(x-e)$.

$$2: \begin{cases} 1: f'(e) \\ 1: \text{equation of tangent line} \end{cases}$$

(b) $f''(x) = x + 2x\ln x$.

For $1 < x < 3$, $x > 0$ and $\ln x > 0$, so $f''(x) > 0$. Thus, the graph of f is concave up on $(1, 3)$.

$$3: \begin{cases} 2: f''(x) \\ 1: \text{answer with reason} \end{cases}$$

(c) Since $f(x) = \int(x^2\ln x)\,dx$, we consider integration by parts.

$$u = \ln x \qquad dv = x^2\,dx$$
$$du = \frac{1}{x}\,dx \qquad v = \int(x^2)\,dx = \frac{1}{3}x^3$$

Therefore,
$$\begin{aligned} f(x) &= \int(x^2\ln x)\,dx \\ &= \frac{1}{3}x^3\ln x - \int\left(\frac{1}{3}x^3\cdot\frac{1}{x}\right)dx \\ &= \frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C. \end{aligned}$$

Since $f(e) = 2$, $2 = \frac{e^3}{3} - \frac{e^3}{9} + C$ and $C = 2 - \frac{2}{9}e^3$.

Thus, $f(x) = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + 2 - \frac{2}{9}e^3$.

$$4: \begin{cases} 2: \text{antiderivative} \\ 1: \text{uses } f(e) = 2 \\ 1: \text{answer} \end{cases}$$

Question 5

(a) $r(5.4) \approx r(5) + r'(5)\Delta t = 30 + 2(0.4) = 30.8$ ft
Since the graph of r is concave down on the interval $5 < t < 5.4$, this estimate is greater than $r(5.4)$.

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{conclusion with reason} \end{cases}$

(b) $\frac{dV}{dt} = 3\left(\frac{4}{3}\right)\pi r^2 \frac{dr}{dt}$

$\left.\frac{dV}{dt}\right|_{t=5} = 4\pi(30)^2 2 = 7200\pi \text{ ft}^3/\text{min}$

$3 : \begin{cases} 2 : \frac{dV}{dt} \\ 1 : \text{answer} \end{cases}$

(c) $\int_0^{12} r'(t)\,dt \approx 2(4.0) + 3(2.0) + 2(1.2) + 4(0.6) + 1(0.5)$
$= 19.3$ ft
$\int_0^{12} r'(t)\,dt$ is the change in the radius, in feet, from $t = 0$ to $t = 12$ minutes.

$2 : \begin{cases} 1 : \text{approximation} \\ 1 : \text{explanation} \end{cases}$

(d) Since r is concave down, r' is decreasing on $0 < t < 12$. Therefore, this approximation, 19.3 ft, is less than $\int_0^{12} r'(t)\,dt$.

1 : conclusion with reason

Units of ft^3/min in part (b) and ft in part (c)

1 : units in (b) and (c)

Question 6

(a) $e^{-x^2} = 1 + \frac{(-x^2)}{1!} + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots + \frac{(-x^2)^n}{n!} + \cdots$

$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots$

$3 : \begin{cases} 1 : \text{two of } 1, -x^2, \frac{x^4}{2}, -\frac{x^6}{6} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

(b) $\frac{1 - x^2 - f(x)}{x^4} = -\frac{1}{2} + \frac{x^2}{6} + \sum_{n=4}^{\infty} \frac{(-1)^{n+1} x^{2n-4}}{n!}$

Thus, $\lim_{x \to 0}\left(\frac{1 - x^2 - f(x)}{x^4}\right) = -\frac{1}{2}$.

1 : answer

(c) $\int_0^x e^{-t^2}\,dt = \int_0^x \left(1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} + \cdots + \frac{(-1)^n t^{2n}}{n!} + \cdots\right) dt$

$= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots$

Using the first two terms of this series, we estimate that $\int_0^{1/2} e^{-t^2}\,dt \approx \frac{1}{2} - \left(\frac{1}{3}\right)\left(\frac{1}{8}\right) = \frac{11}{24}$.

$3 : \begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{estimate} \end{cases}$

(d) $\left|\int_0^{1/2} e^{-t^2}\,dt - \frac{11}{24}\right| < \left(\frac{1}{2}\right)^5 \cdot \frac{1}{10} = \frac{1}{320} < \frac{1}{200}$, since

$\int_0^{1/2} e^{-t^2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+1}}{n!(2n+1)}$, which is an alternating series with individual terms that decrease in absolute value to 0.

$2 : \begin{cases} 1 : \text{uses the third term as the error bound} \\ 1 : \text{explanation} \end{cases}$