BC Calculus
Rubrics
FR 5 and 6

Question 1

(a) $a(4) = \langle x''(4), y''(4) \rangle = \langle 0.433, -11.872 \rangle$

1 : answer

(b) $y(0) = 5 + \int_4^0 3\cos\left(\frac{t^2}{2}\right) dt = 1.600$ or 1.601

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{uses } y(4) = 5 \\ 1 : \text{answer} \end{cases}$

(c) Speed $= \sqrt{(x'(t))^2 + (y'(t))^2}$

$$= \sqrt{3t + 9\cos^2\left(\frac{t^2}{2}\right)} = 3.5$$

The particle first reaches this speed when $t = 2.225$ or 2.226.

$3 : \begin{cases} 1 : \text{ expression for speed} \\ 1 : \text{ equation} \\ 1 : \text{ answer} \end{cases}$

(d) $\int_0^4 \sqrt{3t + 9\cos^2\left(\frac{t^2}{2}\right)}\, dt = 13.182$

$2 : \begin{cases} 1 : \text{ integral} \\ 1 : \text{ answer} \end{cases}$

Question 2

(a) $\int_0^2 r(t)\, dt = 206.370$ kilometers

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\frac{dg}{dt} = \frac{dg}{dx} \cdot \frac{dx}{dt};\quad \frac{dx}{dt} = r(t)$

$$\left.\frac{dg}{dt}\right|_{t=2} = \left.\frac{dg}{dx}\right|_{x=206.370} \cdot r(2)$$
$$= (0.050)(120) = 6 \text{ liters/hour}$$

$3 : \begin{cases} 2 : \text{uses chain rule} \\ 1 : \text{answer with units} \end{cases}$

(c) Let T be the time at which the car's speed reaches 80 kilometers per hour.

Then, $r(T) = 80$ or $T = 0.331453$ hours.

At time T, the car has gone
$x(T) = \int_0^T r(t)\, dt = 10.794097$ kilometers
and has consumed $g(x(T)) = 0.537$ liters of gasoline.

$4 : \begin{cases} 1 : \text{equation } r(t) = 80 \\ 2 : \text{distance integral} \\ 1 : \text{answer} \end{cases}$

Question 3

(a) $\frac{(0+7)}{2}\cdot 8+\frac{(7+8)}{2}\cdot 6+\frac{(8+2)}{2}\cdot 8+\frac{(2+0)}{2}\cdot 2$
$= 115 \text{ ft}^2$

1 : trapezoidal approximation

(b) $\frac{1}{120}\int_0^{120} 115v(t)\,dt$
$= 1807.169$ or $1807.170 \text{ ft}^3/\text{min}$

$3 : \begin{cases} 1 : \text{limits and average value constant} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(c) $\int_0^{24} 8\sin\left(\frac{\pi x}{24}\right) dx = 122.230$ or 122.231 ft^2

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(d) Let C be the cross-sectional area approximation from part (c). The average volumetric flow is
$\frac{1}{20}\int_{40}^{60} C\cdot v(t)\,dt = 2181.912$ or $2181.913 \text{ ft}^3/\text{min}$.

Yes, water must be diverted since the average volumetric flow for this 20-minute period exceeds $2100 \text{ ft}^3/\text{min}$.

$3 : \begin{cases} 1 : \text{volumetric flow integral} \\ 1 : \text{average volumetric flow} \\ 1 : \text{answer with reason} \end{cases}$

Question 4

(a) For $x \ge 0$, $f(x) = x^2(k-x) \ge 0$ if $0 \le x \le k$

$$\int_0^k \left(kx^2 - x^3\right) dx = \left(\frac{k}{3}x^3 - \frac{1}{4}x^4\right)\Bigg|_{x=0}^{x=k} = \frac{k^4}{12}$$

Area $= \frac{k^4}{12} = 2$; $k = \sqrt[4]{24}$

$4 : \begin{cases} 1 : \text{integral} \\ 1 : \text{antiderivative} \\ 1 : \text{value of integral} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi\int_0^k \left(kx^2 - x^3\right)^2 dx$

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and constant} \end{cases}$

(c) Perimeter $= k + \int_0^k \sqrt{1+\left(2kx-3x^2\right)^2}\,dx$

$3 : \begin{cases} 1 : \int_0^k \sqrt{1+(f'(x))^2}\,dx \\ 1 : \text{uses } f'(x) = 2kx - 3x^2 \text{ in integrand} \\ 1 : \text{answer} \end{cases}$

Question 5

(a) g' changes from increasing to decreasing at $x = 1$;
g' changes from decreasing to increasing at $x = 4$.

Points of inflection for the graph of $y = g(x)$ occur at $x = 1$ and $x = 4$.

$2 : \begin{cases} 1 : x\text{-values} \\ 1 : \text{justification} \end{cases}$

(b) The only sign change of g' from positive to negative in the interval is at $x = 2$.

$$g(-3) = 5 + \int_2^{-3} g'(x)\,dx = 5 + \left(-\frac{3}{2}\right) + 4 = \frac{15}{2}$$
$$g(2) = 5$$
$$g(7) = 5 + \int_2^{7} g'(x)\,dx = 5 + (-4) + \frac{1}{2} = \frac{3}{2}$$

The maximum value of g for $-3 \le x \le 7$ is $\frac{15}{2}$.

$3 : \begin{cases} 1 : \text{identifies } x = 2 \text{ as a candidate} \\ 1 : \text{considers endpoints} \\ 1 : \text{maximum value and justification} \end{cases}$

(c) $\dfrac{g(7) - g(-3)}{7 - (-3)} = \dfrac{\frac{3}{2} - \frac{15}{2}}{10} = -\dfrac{3}{5}$

$2 : \begin{cases} 1 : \text{difference quotient} \\ 1 : \text{answer} \end{cases}$

(d) $\dfrac{g'(7) - g'(-3)}{7 - (-3)} = \dfrac{1 - (-4)}{10} = \dfrac{1}{2}$

No, the MVT does *not* guarantee the existence of a value c with the stated properties because g' is not differentiable for at least one point in $-3 < x < 7$.

$2 : \begin{cases} 1 : \text{average value of } g'(x) \\ 1 : \text{answer "No" with reason} \end{cases}$

Question 6

(a) $$\frac{1}{1-u} = 1 + u + u^2 + \cdots + u^n + \cdots$$
$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + \left(-x^2\right)^n + \cdots$$
$$\frac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + \cdots + (-1)^n 2x^{2n+1} + \cdots$$

$3 : \begin{cases} 1 : \text{two of the first four terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

(b) No, the series does not converge when $x = 1$ because when $x = 1$, the terms of the series do not converge to 0.

$1 :$ answer with reason

(c) $$\ln\left(1 + x^2\right) = \int_0^x \frac{2t}{1+t^2}\,dt$$
$$= \int_0^x \left(2t - 2t^3 + 2t^5 - 2t^7 + \cdots\right) dt$$
$$= x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \cdots$$

$2 : \begin{cases} 1 : \text{two of the first four terms} \\ 1 : \text{remaining terms} \end{cases}$

(d) $$\ln\left(\frac{5}{4}\right) = \ln\left(1 + \frac{1}{4}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4 + \frac{1}{3}\left(\frac{1}{2}\right)^6 - \frac{1}{4}\left(\frac{1}{2}\right)^8 + \cdots$$

Let $A = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4 = \frac{7}{32}$.

Since the series is a converging alternating series and the absolute values of the individual terms decrease to 0,

$$\left|A - \ln\left(\frac{5}{4}\right)\right| < \left|\frac{1}{3}\left(\frac{1}{2}\right)^6\right| = \frac{1}{3} \cdot \frac{1}{64} < \frac{1}{100}.$$

$3 : \begin{cases} 1 : \text{uses } x = \frac{1}{2} \\ 1 : \text{value of } A \\ 1 : \text{justification} \end{cases}$