BC Calculus
Rubrics FR1, FR2

Question 1

(a) Area $= 30 \cdot 20 - \int_0^{30} f(x)\,dx = 218.028\text{ cm}^2$

$3 : \begin{cases} 2 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \displaystyle\int_0^{30} \frac{\pi}{2}\left(\frac{f(x)}{2}\right)^2 dx = 2356.194\text{ cm}^3$

Therefore, the baker needs $2356.194 \times 0.05 = 117.809$ or 117.810 grams of chocolate.

$3 : \begin{cases} 2 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(c) Perimeter $= 30 + \int_0^{30} \sqrt{1 + (f'(x))^2}\,dx = 81.803$ or 81.804 cm

$3 : \begin{cases} 2 : \text{integral} \\ 1 : \text{answer} \end{cases}$

Question 2

(a) $35 + \int_0^5 f(t)\,dt = 26.494$ or 26.495 meters

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) Four hours after the storm began, the rate of change of the distance between the road and the edge of the water is increasing at a rate of 1.007 meters/hours^2.

$2 : \begin{cases} 1 : \text{interpretation of } f'(4) \\ 1 : \text{units} \end{cases}$

(c) $f'(t) = 0$ when $t = 0.66187$ and $t = 2.84038$
The minimum of f for $0 \le t \le 5$ may occur at 0, 0.66187, 2.84038, or 5.

$f(0) = -2$
$f(0.66187) = -1.39760$
$f(2.84038) = -2.26963$
$f(5) = -0.48027$

The distance between the road and the edge of the water was decreasing most rapidly at time $t = 2.840$ hours after the storm began.

$3 : \begin{cases} 1 : \text{considers } f'(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(d) $-\int_0^5 f(t)\,dt = \int_0^x g(p)\,dp$

$2 : \begin{cases} 1 : \text{integral of } g \\ 1 : \text{answer} \end{cases}$

Question 3

(a) $\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \frac{2}{3}$

$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} < 0$

Since the one-sided limits do not agree, f is not differentiable at $x = 0$.

2 : $\begin{cases} 1 : \text{sets up difference quotient at } x = 0 \\ 1 : \text{answer with justification} \end{cases}$

(b) $\frac{f(6) - f(a)}{6 - a} = 0$ when $f(a) = f(6)$. There are two values of a for which this is true.

2 : $\begin{cases} 1 : \text{expression for average rate of change} \\ 1 : \text{answer with reason} \end{cases}$

(c) Yes, $a = 3$. The function f is differentiable on the interval $3 < x < 6$ and continuous on $3 \le x \le 6$.

Also, $\frac{f(6) - f(3)}{6 - 3} = \frac{1 - 0}{6 - 3} = \frac{1}{3}$.

By the Mean Value Theorem, there is a value c, $3 < c < 6$, such that $f'(c) = \frac{1}{3}$.

2 : $\begin{cases} 1 : \text{answers “yes” and identifies } a = 3 \\ 1 : \text{justification} \end{cases}$

(d) $g'(x) = f(x)$, $g''(x) = f'(x)$

$g''(x) > 0$ when $f'(x) > 0$

This is true for $-4 < x < 0$ and $3 < x < 6$.

3 : $\begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{considers } g''(x) > 0 \\ 1 : \text{answer} \end{cases}$

Question 4

(a) $r(0) = -1$; $r(\theta) = 0$ when $\theta = \frac{\pi}{3}$.

Area of $S = \frac{1}{2}\int_0^{\pi/3} (1 - 2\cos\theta)^2 \, d\theta$

2 : $\begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \end{cases}$

(b) $x = r\cos\theta$ and $y = r\sin\theta$

$$\frac{dr}{d\theta} = 2\sin\theta$$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta = 4\sin\theta\cos\theta - \sin\theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta = 2\sin^2\theta + (1 - 2\cos\theta)\cos\theta$$

4 : $\begin{cases} 1 : \text{uses } x = r\cos\theta \text{ and } y = r\sin\theta \\ 1 : \frac{dr}{d\theta} \\ 2 : \text{answer} \end{cases}$

(c) When $\theta = \frac{\pi}{2}$, we have $x = 0$, $y = 1$.

$$\left.\frac{dy}{dx}\right|_{\theta = \frac{\pi}{2}} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta = \frac{\pi}{2}} = -2$$

The tangent line is given by $y = 1 - 2x$.

3 : $\begin{cases} 1 : \text{values for } x \text{ and } y \\ 1 : \text{expression for } \frac{dy}{dx} \\ 1 : \text{tangent line equation} \end{cases}$

Question 5

(a) $g(1) = e^{f(1)} = e^2$

$g'(x) = e^{f(x)} f'(x), \; g'(1) = e^{f(1)} f'(1) = -4e^2$

The tangent line is given by $y = e^2 - 4e^2(x-1)$.

$3: \begin{cases} 1: g'(x) \\ 1: g(1) \text{ and } g'(1) \\ 1: \text{tangent line equation} \end{cases}$

(b) $g'(x) = e^{f(x)} f'(x)$

$e^{f(x)} > 0$ for all x

So, g' changes from positive to negative only when f' changes from positive to negative. This occurs at $x = -1$ only. Thus, g has a local maximum at $x = -1$.

$2: \begin{cases} 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(c) $g''(-1) = e^{f(-1)}\left[(f'(-1))^2 + f''(-1)\right]$

$e^{f(-1)} > 0$ and $f'(-1) = 0$

Since f' is decreasing on a neighborhood of -1, $f''(-1) < 0$. Therefore, $g''(-1) < 0$.

$2: \begin{cases} 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(d) $\dfrac{g'(3) - g'(1)}{3-1} = \dfrac{e^{f(3)} f'(3) - e^{f(1)} f'(1)}{2} = 2e^2$

$2: \begin{cases} 1: \text{difference quotient} \\ 1: \text{answer} \end{cases}$

Question 6

(a) The power series is geometric with ratio $(x+1)$.

The series converges if and only if $|x+1| < 1$.

Therefore, the interval of convergence is $-2 < x < 0$.

$3: \begin{cases} 1: \text{identifies as geometric} \\ 1: |x+1| < 1 \\ 1: \text{interval of convergence} \end{cases}$

OR

$\lim\limits_{n\to\infty} \left|\dfrac{(x+1)^{n+1}}{(x+1)^n}\right| = |x+1| < 1$ when $-2 < x < 0$

At $x = -2$, the series is $\sum\limits_{n=0}^{\infty} (-1)^n$, which diverges since the terms do not converge to 0. At $x = 0$, the series is $\sum\limits_{n=0}^{\infty} 1$, which similarly diverges. Therefore, the interval of convergence is $-2 < x < 0$.

OR

$3: \begin{cases} 1: \text{sets up limit of ratio} \\ 1: \text{radius of convergence} \\ 1: \text{interval of convergence} \end{cases}$

(b) Since the series is geometric,

$f(x) = \sum\limits_{n=0}^{\infty} (x+1)^n = \dfrac{1}{1-(x+1)} = -\dfrac{1}{x}$ for $-2 < x < 0$.

1 : answer

(c) $g\left(-\frac{1}{2}\right) = \int_{-1}^{-\frac{1}{2}} -\frac{1}{x}\, dx = -\ln|x|\Big|_{x=-1}^{x=-\frac{1}{2}} = \ln 2$

$2: \begin{cases} 1: \text{antiderivative} \\ 1: \text{value} \end{cases}$

(d) $h(x) = f(x^2 - 1) = 1 + x^2 + x^4 + \cdots + x^{2n} + \cdots$

$h\left(\frac{1}{2}\right) = f\left(-\frac{3}{4}\right) = \frac{4}{3}$

$3: \begin{cases} 1: \text{first three terms} \\ 1: \text{general term} \\ 1: \text{value of } h\left(\frac{1}{2}\right) \end{cases}$