AB Calculus
Rubrics
FR 7 & 8

Question1

$e^{2x-x^2} = 2$ when $x = 0.446057, 1.553943$
Let $P = 0.446057$ and $Q = 1.553943$

(a) Area of $R = \int_P^Q \left(e^{2x-x^2} - 2\right) dx = 0.514$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$

(b) $e^{2x-x^2} = 1$ when $x = 0, 2$

Area of $S = \int_0^2 \left(e^{2x-x^2} - 1\right) dx -$ Area of R
$= 2.06016 -$ Area of $R = 1.546$

OR

$\int_0^P \left(e^{2x-x^2} - 1\right) dx + (Q - P)\cdot 1 + \int_Q^2 \left(e^{2x-x^2} - 1\right) dx$
$= 0.219064 + 1.107886 + 0.219064 = 1.546$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$

(c) Volume $= \pi \int_P^Q \left(\left(e^{2x-x^2} - 1\right)^2 - (2-1)^2 \right) dx$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{constant and limits} \end{cases}$

Question 2

(a) $a(3) = v'(3) = 6\cos 9 = -5.466$ or -5.467

$1 : a(3)$

(b) Distance $= \int_0^3 |v(t)|\, dt = 1.702$

OR

For $0 < t < 3$, $v(t) = 0$ when $t = \sqrt{\pi} = 1.77245$ and
$t = \sqrt{2\pi} = 2.50663$
$x(0) = 5$
$x(\sqrt{\pi}) = 5 + \int_0^{\sqrt{\pi}} v(t)\, dt = 5.89483$
$x(\sqrt{2\pi}) = 5 + \int_0^{\sqrt{2\pi}} v(t)\, dt = 5.43041$
$x(3) = 5 + \int_0^3 v(t)\, dt = 5.77356$
$\left|x(\sqrt{\pi}) - x(0)\right| + \left|x(\sqrt{2\pi}) - x(\sqrt{\pi})\right| + \left|x(3) - x(\sqrt{2\pi})\right| = 1.702$

$2 : \begin{cases} 1 : \text{setup} \\ 1 : \text{answer} \end{cases}$

(c) $x(3) = 5 + \int_0^3 v(t)\, dt = 5.773$ or 5.774

$3 : \begin{cases} 2 \begin{cases} 1 : \text{integrand} \\ 1 : \text{uses } x(0) = 5 \end{cases} \\ 1 : \text{answer} \end{cases}$

(d) The particle's rightmost position occurs at time $t = \sqrt{\pi} = 1.772$.

The particle changes from moving right to moving left at those times t for which $v(t) = 0$ with $v(t)$ changing from positive to negative, namely at $t = \sqrt{\pi}, \sqrt{3\pi}, \sqrt{5\pi}$ ($t = 1.772, 3.070, 3.963$).

Using $x(T) = 5 + \int_0^T v(t)\, dt$, the particle's positions at the times it changes from rightward to leftward movement are:

T:	0	$\sqrt{\pi}$	$\sqrt{3\pi}$	$\sqrt{5\pi}$
$x(T)$:	5	5.895	5.788	5.752

The particle is farthest to the right when $T = \sqrt{\pi}$.

$3 : \begin{cases} 1 : \text{sets } v(t) = 0 \\ 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$

Question 3

(a) $W'(20) = -22.1 \cdot 0.16 \cdot 20^{-0.84} = -0.285$ or -0.286

When $v = 20$ mph, the wind chill is decreasing at 0.286 °F/mph.

$2 : \begin{cases} 1 : \text{value} \\ 1 : \text{explanation} \end{cases}$

(b) The average rate of change of W over the interval $5 \le v \le 60$ is $\dfrac{W(60) - W(5)}{60 - 5} = -0.253$ or -0.254.

$W'(v) = \dfrac{W(60) - W(5)}{60 - 5}$ when $v = 23.011$.

$3 : \begin{cases} 1 : \text{average rate of change} \\ 1 : W'(v) = \text{average rate of change} \\ 1 : \text{value of } v \end{cases}$

(c) $\left.\dfrac{dW}{dt}\right|_{t=3} = \left.\left(\dfrac{dW}{dv} \cdot \dfrac{dv}{dt}\right)\right|_{t=3} = W'(35) \cdot 5 = -0.892$ °F/hr

OR

$W = 55.6 - 22.1(20 + 5t)^{0.16}$

$\left.\dfrac{dW}{dt}\right|_{t=3} = -0.892$ °F/hr

$3 : \begin{cases} 1 : \dfrac{dv}{dt} = 5 \\ 1 : \text{uses } v(3) = 35, \\ \quad\quad \text{or} \\ \quad \text{uses } v(t) = 20 + 5t \\ 1 : \text{answer} \end{cases}$

Units of °F/mph in (a) and °F/hr in (c)

1 : units in (a) and (c)

Question 4

(a) $f'(x) = 0$ at $x = -3, 1, 4$

f' changes from positive to negative at -3 and 4.

Thus, f has a relative maximum at $x = -3$ and at $x = 4$.

$2 : \begin{cases} 1 : x\text{-values} \\ 1 : \text{justification} \end{cases}$

(b) f' changes from increasing to decreasing, or vice versa, at $x = -4, -1$, and 2. Thus, the graph of f has points of inflection when $x = -4, -1$, and 2.

$2 : \begin{cases} 1 : x\text{-values} \\ 1 : \text{justification} \end{cases}$

(c) The graph of f is concave up with positive slope where f' is increasing and positive: $-5 < x < -4$ and $1 < x < 2$.

$2 : \begin{cases} 1 : \text{intervals} \\ 1 : \text{explanation} \end{cases}$

(d) Candidates for the absolute minimum are where f' changes from negative to positive (at $x = 1$) and at the endpoints ($x = -5, 5$).

$f(-5) = 3 + \int_1^{-5} f'(x)\,dx = 3 - \frac{\pi}{2} + 2\pi > 3$

$f(1) = 3$

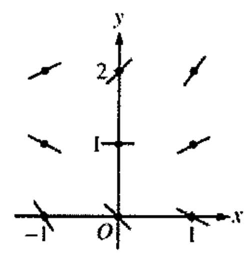
$f(5) = 3 + \int_1^5 f'(x)\,dx = 3 + \frac{3 \cdot 2}{2} - \frac{1}{2} > 3$

The absolute minimum value of f on $[-5, 5]$ is $f(1) = 3$.

$3 : \begin{cases} 1 : \text{identifies } x = 1 \text{ as a candidate} \\ 1 : \text{considers endpoints} \\ 1 : \text{value and explanation} \end{cases}$

Question 5

(a)



2 : Sign of slope at each point and relative steepness of slope lines in rows and columns.

(b) $\dfrac{d^2y}{dx^2} = \dfrac{1}{2} + \dfrac{dy}{dx} = \dfrac{1}{2}x + y - \dfrac{1}{2}$

Solution curves will be concave up on the half-plane above the line $y = -\dfrac{1}{2}x + \dfrac{1}{2}$.

$3 : \begin{cases} 2 : \dfrac{d^2y}{dx^2} \\ 1 : \text{description} \end{cases}$

(c) $\left.\dfrac{dy}{dx}\right|_{(0,\,1)} = 0 + 1 - 1 = 0$ and $\left.\dfrac{d^2y}{dx^2}\right|_{(0,\,1)} = 0 + 1 - \dfrac{1}{2} > 0$

Thus, f has a relative minimum at $(0, 1)$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

(d) Substituting $y = mx + b$ into the differential equation:

$m = \dfrac{1}{2}x + (mx + b) - 1 = \left(m + \dfrac{1}{2}\right)x + (b - 1)$

Then $0 = m + \dfrac{1}{2}$ and $m = b - 1$: $m = -\dfrac{1}{2}$ and $b = \dfrac{1}{2}$.

$2 : \begin{cases} 1 : \text{value for } m \\ 1 : \text{value for } b \end{cases}$

Question 6

(a) The Mean Value Theorem guarantees that there is a value c, with $2 < c < 5$, so that

$f'(c) = \dfrac{f(5) - f(2)}{5 - 2} = \dfrac{2 - 5}{5 - 2} = -1.$

$2 : \begin{cases} 1 : \dfrac{f(5) - f(2)}{5 - 2} \\ 1 : \text{conclusion, using MVT} \end{cases}$

(b) $g'(x) = f'(f(x)) \cdot f'(x)$

$g'(2) = f'(f(2)) \cdot f'(2) = f'(5) \cdot f'(2)$

$g'(5) = f'(f(5)) \cdot f'(5) = f'(2) \cdot f'(5)$

Thus, $g'(2) = g'(5)$.

Since f is twice-differentiable, g' is differentiable everywhere, so the Mean Value Theorem applied to g' on $[2, 5]$ guarantees there is a value k, with $2 < k < 5$, such that $g''(k) = \dfrac{g'(5) - g'(2)}{5 - 2} = 0$.

$3 : \begin{cases} 1 : g'(x) \\ 1 : g'(2) = f'(5) \cdot f'(2) = g'(5) \\ 1 : \text{uses MVT with } g' \end{cases}$

(c) $g''(x) = f''(f(x)) \cdot f'(x) \cdot f'(x) + f'(f(x)) \cdot f''(x)$

If $f''(x) = 0$ for all x, then

$g''(x) = 0 \cdot f'(x) \cdot f'(x) + f'(f(x)) \cdot 0 = 0$ for all x.

Thus, there is no x-value at which $g''(x)$ changes sign, so the graph of g has no inflection points.

OR

If $f''(x) = 0$ for all x, then f is linear, so $g = f \circ f$ is linear and the graph of g has no inflection points.

$2 : \begin{cases} 1 : \text{considers } g'' \\ 1 : g''(x) = 0 \text{ for all } x \end{cases}$

OR

$2 : \begin{cases} 1 : f \text{ is linear} \\ 1 : g \text{ is linear} \end{cases}$

(d) Let $h(x) = f(x) - x$.

$h(2) = f(2) - 2 = 5 - 2 = 3$

$h(5) = f(5) - 5 = 2 - 5 = -3$

Since $h(2) > 0 > h(5)$, the Intermediate Value Theorem guarantees that there is a value r, with $2 < r < 5$, such that $h(r) = 0$.

$2 : \begin{cases} 1 : h(2) \text{ and } h(5) \\ 1 : \text{conclusion, using IVT} \end{cases}$