AB Calculus
Rubrics
FR 5 and 6

Question 1

The graphs of $y = \sqrt{x}$ and $y = \frac{x}{3}$ intersect at the points $(0, 0)$ and $(9, 3)$.

(a) $\int_0^9 \left(\sqrt{x} - \frac{x}{3}\right) dx = 4.5$

OR

$\int_0^3 \left(3y - y^2\right) dy = 4.5$

$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) $\pi \int_0^3 \left((3y+1)^2 - \left(y^2+1\right)^2\right) dy$

$= \frac{207\pi}{5} = 130.061$ or 130.062

$2\pi \int_0^9 (x+1)\left(\sqrt{x} - \frac{x}{3}\right) dx$

$4 : \begin{cases} 1 : \text{constant and limits} \\ 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(c) $\int_0^3 \left(3y - y^2\right)^2 dy = 8.1$

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and answer} \end{cases}$

Question 2

(a) $\int_0^2 r(t)\, dt = 206.370$ kilometers

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $\frac{dg}{dt} = \frac{dg}{dx} \cdot \frac{dx}{dt}; \quad \frac{dx}{dt} = r(t)$

$\left.\frac{dg}{dt}\right|_{t=2} = \left.\frac{dg}{dx}\right|_{x=206.370} \cdot r(2)$

$= (0.050)(120) = 6$ liters/hour

$3 : \begin{cases} 2 : \text{uses chain rule} \\ 1 : \text{answer with units} \end{cases}$

(c) Let T be the time at which the car's speed reaches 80 kilometers per hour.

Then, $r(T) = 80$ or $T = 0.331453$ hours.

At time T, the car has gone

$x(T) = \int_0^T r(t)\, dt = 10.794097$ kilometers

and has consumed $g(x(T)) = 0.537$ liters of gasoline.

$4 : \begin{cases} 1 : \text{equation } r(t) = 80 \\ 2 : \text{distance integral} \\ 1 : \text{answer} \end{cases}$

Question 3

(a) $\dfrac{(0+7)}{2}\cdot 8+\dfrac{(7+8)}{2}\cdot 6+\dfrac{(8+2)}{2}\cdot 8+\dfrac{(2+0)}{2}\cdot 2$
$= 115 \text{ ft}^2$

1 : trapezoidal approximation

(b) $\dfrac{1}{120}\displaystyle\int_0^{120} 115v(t)\,dt$
$= 1807.169$ or $1807.170 \text{ ft}^3/\text{min}$

$3:\begin{cases}1: \text{limits and average value constant}\\ 1: \text{integrand}\\ 1: \text{answer}\end{cases}$

(c) $\displaystyle\int_0^{24} 8\sin\left(\frac{\pi x}{24}\right)dx = 122.230$ or 122.231 ft^2

$2:\begin{cases}1: \text{integral}\\ 1: \text{answer}\end{cases}$

(d) Let C be the cross-sectional area approximation from part (c). The average volumetric flow is
$\dfrac{1}{20}\displaystyle\int_{40}^{60} C\cdot v(t)\,dt = 2181.912$ or $2181.913 \text{ ft}^3/\text{min}.$

Yes, water must be diverted since the average volumetric flow for this 20-minute period exceeds $2100 \text{ ft}^3/\text{min}$.

$3:\begin{cases}1: \text{volumetric flow integral}\\ 1: \text{average volumetric flow}\\ 1: \text{answer with reason}\end{cases}$

Question 4

(a) $f'(x) = 3\sqrt{4+(3x)^2}$

$g'(x) = f'(\sin x)\cdot\cos x$

$= 3\sqrt{4+(3\sin x)^2}\cdot\cos x$

$4:\begin{cases}2: f'(x)\\ 2: g'(x)\end{cases}$

(b) $g(\pi)=0,\ g'(\pi)=-6$
Tangent line: $y=-6(x-\pi)$

$2:\begin{cases}1: g(\pi) \text{ or } g'(\pi)\\ 1: \text{tangent line equation}\end{cases}$

(c) For $0<x<\pi$, $g'(x)=0$ only at $x=\dfrac{\pi}{2}$.

$g(0)=g(\pi)=0$

$g\left(\dfrac{\pi}{2}\right)=\displaystyle\int_0^3\sqrt{4+t^2}\,dt>0$

The maximum value of g on $[0,\pi]$ is
$\displaystyle\int_0^3\sqrt{4+t^2}\,dt.$

$3:\begin{cases}1: \text{sets } g'(x)=0\\ 1: \text{justifies maximum at } \frac{\pi}{2}\\ 1: \text{integral expression for } g\left(\frac{\pi}{2}\right)\end{cases}$

Question 5

(a) g' changes from increasing to decreasing at $x = 1$;
g' changes from decreasing to increasing at $x = 4$.

Points of inflection for the graph of $y = g(x)$ occur at $x = 1$ and $x = 4$.

$2: \begin{cases} 1: x\text{-values} \\ 1: \text{justification} \end{cases}$

(b) The only sign change of g' from positive to negative in the interval is at $x = 2$.

$$g(-3) = 5 + \int_2^{-3} g'(x)\,dx = 5 + \left(-\frac{3}{2}\right) + 4 = \frac{15}{2}$$
$$g(2) = 5$$
$$g(7) = 5 + \int_2^{7} g'(x)\,dx = 5 + (-4) + \frac{1}{2} = \frac{3}{2}$$

The maximum value of g for $-3 \le x \le 7$ is $\frac{15}{2}$.

$3: \begin{cases} 1: \text{identifies } x = 2 \text{ as a candidate} \\ 1: \text{considers endpoints} \\ 1: \text{maximum value and justification} \end{cases}$

(c) $\dfrac{g(7) - g(-3)}{7 - (-3)} = \dfrac{\frac{3}{2} - \frac{15}{2}}{10} = -\dfrac{3}{5}$

$2: \begin{cases} 1: \text{difference quotient} \\ 1: \text{answer} \end{cases}$

(d) $\dfrac{g'(7) - g'(-3)}{7 - (-3)} = \dfrac{1 - (-4)}{10} = \dfrac{1}{2}$

No, the MVT does *not* guarantee the existence of a value c with the stated properties because g' is not differentiable for at least one point in $-3 < x < 7$.

$2: \begin{cases} 1: \text{average value of } g'(x) \\ 1: \text{answer "No" with reason} \end{cases}$

Question 6

(a) $2x + 2 + 4y^3\dfrac{dy}{dx} + 4\dfrac{dy}{dx} = 0$

$(4y^3 + 4)\dfrac{dy}{dx} = -2x - 2$

$\dfrac{dy}{dx} = \dfrac{-2(x+1)}{4(y^3+1)} = \dfrac{-(x+1)}{2(y^3+1)}$

$2: \begin{cases} 1: \text{implicit differentiation} \\ 1: \text{verification} \end{cases}$

(b) $\left.\dfrac{dy}{dx}\right|_{(-2,1)} = \dfrac{-(-2+1)}{2(1+1)} = \dfrac{1}{4}$

Tangent line: $y = 1 + \frac{1}{4}(x + 2)$

$2: \begin{cases} 1: \text{slope} \\ 1: \text{tangent line equation} \end{cases}$

(c) Vertical tangent lines occur at points on the curve where $y^3 + 1 = 0$ (or $y = -1$) and $x \ne -1$.

On the curve, $y = -1$ implies that $x^2 + 2x + 1 - 4 = 5$, so $x = -4$ or $x = 2$.

Vertical tangent lines occur at the points $(-4, -1)$ and $(2, -1)$.

$3: \begin{cases} 1: y = -1 \\ 1: \text{substitutes } y = -1 \text{ into the equation of the curve} \\ 1: \text{answer} \end{cases}$

(d) Horizontal tangents occur at points on the curve where $x = -1$ and $y \ne -1$.

The curve crosses the x-axis where $y = 0$.

$(-1)^2 + 2(-1) + 0^4 + 4 \cdot 0 \ne 5$

No, the curve cannot have a horizontal tangent where it crosses the x-axis.

$2: \begin{cases} 1: \text{works with } x = -1 \text{ or } y = 0 \\ 1: \text{answer with reason} \end{cases}$