AB Calculus
Rubrics FR1, FR2

Question 1

(a) $R(t) = 6 + \int_0^t \frac{1}{16}\left(3+\sin\left(x^2\right)\right)dx$ $R(3) = 6.610$ or 6.611	$3: \begin{cases} 1: \text{integral} \\ 1: \text{expression for } R(t) \\ 1: R(3) \end{cases}$
(b) $A(t) = \pi(R(t))^2$ $A'(t) = 2\pi R(t)R'(t)$ $A'(3) = 8.858\ \text{cm}^2/\text{year}$	$3: \begin{cases} 1: \text{expression for } A(t) \\ 1: \text{expression for } A'(t) \\ 1: \text{answer with units} \end{cases}$
(c) $\int_0^3 A'(t)\,dt = A(3) - A(0) = 24.200$ or 24.201 From time $t = 0$ to $t = 3$ years, the cross-sectional area grows by 24.201 square centimeters.	$3: \begin{cases} 1: \text{uses Fundamental Theorem of Calculus} \\ 1: \text{value of } \int_0^3 A'(t)\,dt \\ 1: \text{meaning of } \int_0^3 A'(t)\,dt \end{cases}$

Question 2

(a) $35 + \int_0^5 f(t)\,dt = 26.494$ or 26.495 meters	$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$
(b) Four hours after the storm began, the rate of change of the distance between the road and the edge of the water is increasing at a rate of 1.007 meters/hours^2.	$2: \begin{cases} 1: \text{interpretation of } f'(4) \\ 1: \text{units} \end{cases}$
(c) $f'(t) = 0$ when $t = 0.66187$ and $t = 2.84038$ The minimum of f for $0 \le t \le 5$ may occur at 0, 0.66187, 2.84038, or 5. $f(0) = -2$ $f(0.66187) = -1.39760$ $f(2.84038) = -2.26963$ $f(5) = -0.48027$ The distance between the road and the edge of the water was decreasing most rapidly at time $t = 2.840$ hours after the storm began.	$3: \begin{cases} 1: \text{considers } f'(t) = 0 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$
(d) $-\int_0^5 f(t)\,dt = \int_0^x g(p)\,dp$	$2: \begin{cases} 1: \text{integral of } g \\ 1: \text{answer} \end{cases}$

Question 3

(a) $\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \frac{2}{3}$

$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} < 0$

Since the one-sided limits do not agree, f is not differentiable at $x = 0$.

$2 : \begin{cases} 1 : \text{sets up difference quotient at } x = 0 \\ 1 : \text{answer with justification} \end{cases}$

(b) $\frac{f(6) - f(a)}{6 - a} = 0$ when $f(a) = f(6)$. There are two values of a for which this is true.

$2 : \begin{cases} 1 : \text{expression for average rate of change} \\ 1 : \text{answer with reason} \end{cases}$

(c) Yes, $a = 3$. The function f is differentiable on the interval $3 < x < 6$ and continuous on $3 \le x \le 6$.

Also, $\frac{f(6) - f(3)}{6 - 3} = \frac{1 - 0}{6 - 3} = \frac{1}{3}$.

By the Mean Value Theorem, there is a value c, $3 < c < 6$, such that $f'(c) = \frac{1}{3}$.

$2 : \begin{cases} 1 : \text{answers “yes” and identifies } a = 3 \\ 1 : \text{justification} \end{cases}$

(d) $g'(x) = f(x)$, $g''(x) = f'(x)$

$g''(x) > 0$ when $f'(x) > 0$

This is true for $-4 < x < 0$ and $3 < x < 6$.

$3 : \begin{cases} 1 : g'(x) = f(x) \\ 1 : \text{considers } g''(x) > 0 \\ 1 : \text{answer} \end{cases}$

Question 4

(a) Area $= \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx = \frac{2}{3}x^{3/2} - \frac{x^2}{4}\Big|_{x=0}^{x=4} = \frac{4}{3}$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)^2 dx = \int_0^4 \left(x - x^{3/2} + \frac{x^2}{4}\right) dx$

$= \frac{x^2}{2} - \frac{2x^{5/2}}{5} + \frac{x^3}{12}\Big|_{x=0}^{x=4} = \frac{8}{15}$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) Volume $= \pi \int_0^4 \left(\left(2 - \frac{x}{2}\right)^2 - \left(2 - \sqrt{x}\right)^2\right) dx$

$3 : \begin{cases} 1 : \text{limits and constant} \\ 2 : \text{integrand} \end{cases}$

Question 5

(a) $g(1) = e^{f(1)} = e^2$

$g'(x) = e^{f(x)} f'(x),\ \ g'(1) = e^{f(1)} f'(1) = -4e^2$

The tangent line is given by $y = e^2 - 4e^2(x-1)$.

$3: \begin{cases} 1: g'(x) \\ 1: g(1) \text{ and } g'(1) \\ 1: \text{tangent line equation} \end{cases}$

(b) $g'(x) = e^{f(x)} f'(x)$

$e^{f(x)} > 0$ for all x

So, g' changes from positive to negative only when f' changes from positive to negative. This occurs at $x = -1$ only. Thus, g has a local maximum at $x = -1$.

$2: \begin{cases} 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(c) $g''(-1) = e^{f(-1)}\left[(f'(-1))^2 + f''(-1)\right]$

$e^{f(-1)} > 0$ and $f'(-1) = 0$

Since f' is decreasing on a neighborhood of -1, $f''(-1) < 0$. Therefore, $g''(-1) < 0$.

$2: \begin{cases} 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(d) $\dfrac{g'(3) - g'(1)}{3-1} = \dfrac{e^{f(3)} f'(3) - e^{f(1)} f'(1)}{2} = 2e^2$

$2: \begin{cases} 1: \text{difference quotient} \\ 1: \text{answer} \end{cases}$

Question 6

1 : units in (a) and (b)

(a) $a(36) = v'(36) \approx \dfrac{v(40) - v(32)}{40-32} = \dfrac{11}{8}$ meters/sec^2

1 : answer

(b) $\int_{20}^{40} v(t)\,dt$ is the particle's change in position in meters from time $t = 20$ seconds to time $t = 40$ seconds.

$$\int_{20}^{40} v(t)\,dt \approx \frac{v(20)+v(25)}{2}\cdot 5 + \frac{v(25)+v(32)}{2}\cdot 7 + \frac{v(32)+v(40)}{2}\cdot 8$$
$$= -75 \text{ meters}$$

$3: \begin{cases} 1: \text{meaning of } \int_{20}^{40} v(t)\,dt \\ 2: \text{trapezoidal approximation} \end{cases}$

(c) $v(8) > 0$ and $v(20) < 0$

$v(32) < 0$ and $v(40) > 0$

Therefore, the particle changes direction in the intervals $8 < t < 20$ and $32 < t < 40$.

$2: \begin{cases} 1: \text{answer} \\ 1: \text{explanation} \end{cases}$

(d) Since $v'(t) = a(t) > 0$ for $0 < t < 8$, $v(t) \geq 3$ on this interval.

Therefore, $x(8) = x(0) + \int_0^8 v(t)\,dt \geq 7 + 8\cdot 3 > 30$.

$2: \begin{cases} 1: v'(t) = a(t) \\ 1: \text{explanation of } x(8) > 30 \end{cases}$